SCHATTEN-VON NEUMANN PROPERTIES OF WEYL OPERATORS OF HÖRMANDER TYPE

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Let $t \in \mathbf{R}$ be fixed and consider the *pseudo-differential operators* $\operatorname{Op}_t(a)$ with *symbol* a which is defined by the formula:

$$\operatorname{Op}_t(a)f(x) \equiv (2\pi)^{-n} \iint_{\mathbf{R}^n\times\mathbf{R}^n} a((1-t)x+ty,\xi)f(y)e^{i\langle x-y,\xi\rangle}\, dyd\xi$$

A fundamental result for such operators reads: Assume that $0 \le \delta < \rho \le 1$ and $r \in \mathbf{R}$. Then each $\operatorname{Op}_t(a)$ with $a \in S^r_{\rho,\delta}(\mathbf{R}^{2n})$ is L^2-continuous, if and only if $S^r_{\rho,\delta} \subseteq L^\infty$ (i.e. $r \le 0$). Here recall that $S^r_{\rho,\delta}(\mathbf{R}^{2n})$ consists of all $a \in C^\infty(\mathbf{R}^{2n})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{r-\rho|\beta|+\delta|\alpha|}.$$

A somewhat weak property here is that no conclusion concerning L^2-continuity can be done for a *particular* operator $\operatorname{Op}_t(a)$, when $a \in S^r_{\rho,\delta}$ and $r > 0$.

In a joint paper with E. Buzano, we complete the theory at this point. More precisely, if $a \in S^r_{\rho,\delta}$, then we prove that $\operatorname{Op}_t(a)$ is L^2-continuous, if and only if $a \in L^\infty$.

The theory, which contains the latter result as a special case, is formulated by means of Hörmander-Weyl calculus, where the symbol classes $S(m,g)$ are parameterized with appropriate weight functions m and Riemannian metrics g. The continuity investigations are also performed in a broader context, which involve Schatten-von Neumann properties for such operators. Then we prove the following general result: Assume that $p \in [1,\infty]$, and that the g-Planck's constant h_g satisfies $h_g^N m \in L^p$, for some $N \ge 0$. Then $\operatorname{Op}_t(a)$ is a Schatten-von Neumann operator of order p, if and only if $a \in L^p$.

An important example concerns globally defined pseudo-differential operators with symbols in the SG class $\mathrm{SG}^{(\omega)}_{\rho,\delta}(\mathbf{R}^{2n})$, which consists of all $a \in C^\infty(\mathbf{R}^{2n})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\omega(x,\xi)(1+|x|)^{-\delta|\alpha|}(1+|\xi|)^{-\rho|\beta|},$$

where ω is bounded by a polynomial and $\rho,\delta > 0$. In this case we have that $\operatorname{Op}_t(a)$ is Schatten-p operator, if and only if $a \in L^p$.

In the talk we explain these results and give some ideas of their proofs.

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